Semi Hollow-Lifting Modules

Módulos de elevación semi hueca

**ABSTRACT***/ Let T be a unitary left Module over a ring with identity R. In this paper, we introduced the concept of semihollow-lifting Module as follows, an ℜ-Module T is called semihollow-lifting if for every Submodule H of T with \( T_H \neq 0 \) hollow, there exists a Submodule F of T such that T = F ⊕ F H H and F \( \subseteq \) H in T. Also, study the basic properties of semihollow-lifting Modules. **Keywords**: lifting Modules, hollow Modules. Hollow lifting Modules

**RESUMEN**/ dejemos que T sea un Módulo izquierdo unitario sobre un anillo con identidad R. En este artículo, presentamos el concepto de Módulo de levantamiento de semihollow de la siguiente manera, un Módulo R T se llama levantamiento de semihollow si para cada Submódulo H de T con T / Hueco H, existe un Submódulo F de T tal que T = F \( \oplus \) F H H y F \( \subseteq \) H en T. Además, estudie las propiedades básicas de los módulos de elevación semi-hueco. Palabras clave: modulos de elevación, módulos huecos. Módulos de elevación huecos

**Introduction**

A Submodule H of an ℜ-Module T is called a small Submodule of T (H \( \ll \) T) if for any Submodule F of T such that T = H \( \oplus \) F, then F = T[1]. Let T be an ℜ-Module and let F, H be Submodules of T such that F \( \subseteq \) H \( \subseteq \) T, F is called coessential Submodule of H in T (F \( \subseteq \) sce H in T) if \( H_F \ll T_F \) [2]. An ℜ-Module T is called a hollow-lifting Module if for every Submodule H of T with \( T_H \neq 0 \) hollow, has a coessential Submodule in T that is a direct summand of T [3]. In this paper, we introduced the concept of semihollow-lifting Modules and investigate some properties of semihollow-lifting Modules.

A proper Submodule H of an ℜ-Module T is called semismall of T (H \( \ll_s \) T) if H = 0 or H/F \( \ll \) T/F for all nonzero Submodule F of H [4].

Let T be an ℜ-Module and let H, F be Submodules of T such that F \( \subseteq \) H \( \subseteq \) T. F is called semicoessential Submodule of H in T (F \( \subseteq \) sce H in T) if \( H_F \ll T_F \) [5].

An ℜ-Module T is called semihollow-lifting if for every Submodule H of T with \( T_H \neq 0 \) hollow, there exists a Submodule F of T such that T = F \( \oplus \) F H H and F \( \subseteq \) sce H in T.

A nontrivial ℜ-Module T is called semihollow if every proper Submodule of T is semismall in T [4].

An ℜ-Module T is said to have a semihollow factor Module if there exists a Submodule H of T such that \( T_H \neq 0 \) is semihollow Module [5].

An ℜ-Module T is called a semilifting Module if for every Submodule H of T, there
exists a direct summand \( F \) of \( T \) such that \( F \leq \text{sce} \ H \) in \( T[6] \).

**Examples and Remarks**

1. It is clear that \( Z_4 \) as \( Z \)-Module is semihollow-lifting.
2. Every Module which has no semihollow factor Module is semihollow-lifting.
3. \( Z \) as \( Z \)-Module is not semihollow-lifting. To show that, assume that \( Z \) is semihollow-lifting. Consider the Submodule \( 4Z \). Since \( \frac{Z}{4Z} \) is hollow, there is a direct summand \( H \) of \( Z \) such that \( H \leq \text{sce} \ 4Z \) in \( Z \). But \( Z \) is indecomposable, so \( H = 0 \) and hence \( 4Z \triangleleft Z \) which is a contradiction.
4. Every semilifting Module is semihollow-lifting, in particular every semisimple or semihollow Module is semihollow-lifting. For example \( Z_p \) as \( Z \)-Module, where \( p \) is a prime number. The converse is not true in general. See Example 8.

**Proposition 2** Let \( T \) be an \( \mathbb{R} \)-Module. \( T = F_1 \oplus F_2 \) where \( F_1 \) and \( F_2 \) be hollow Modules. Then the following are equivalent:

1. \( T \) is semihollow lifting Module.
2. \( T \) is semilifting Module.

**Proof:** Let \( H \) be a Submodule of \( T \). Consider the two natural projections maps \( \pi_1:T \to F_1 \) and \( \pi_2:T \to F_2 \). If \( \pi_1(H) \neq F_1 \) and \( \pi_2(H) \neq F_2 \), then \( \pi_1(H) \leq F_1 \) and \( \pi_2(H) \leq F_2 \). Thus, by [6] we get, \( \pi_1(H) \oplus \pi_2(H) \leq F_1 \oplus F_2 \). Claim that \( H \leq \pi_1(H) \oplus \pi_2(H) \), to show that, let \( h \in H \) then \( h \in T \). Thus, \( F_1 \oplus F_2 \) and hence \( h = (f_1, f_2) \), where \( f_1 \in F_1 \) and \( f_2 \in F_2 \). Now, \( \pi_1(h) = \pi_1(f_1, f_2) = f_1 \) and \( \pi_2(h) = \pi_2(f_1, f_2) = f_2 \). This implies that \( h = (\pi_1(h), \pi_2(h)) \) and we get \( H \leq \pi_1(H) \oplus \pi_2(H) \) which implies \( H \leq T \). Thus \( T \) is semilifting Module. Now, \( \pi_1(H) = H_1 \), then \( \pi_1(H) = \pi_1(T) \). It is clear that \( T = H + F_2 \). By the (second isomorphism theorem), \( H + F_2 \cong \frac{F_2}{H \cap F_2} \). Since \( F_2 \) is hollow Module, then \( \frac{F_2}{H \cap F_2} \) is hollow and hence \( T \) is hollow. But \( T \) is semihollow-lifting, therefore there exists a semisoessential Submodule of \( H \) in \( T \) which is a direct summand of \( T \). Thus \( T \) is semihollow-lifting. 

2. Consider the Module \( T = Z_2 \oplus Z_8 \), it is clear that, \( Z_2 \) and \( Z_8 \) as \( Z \)-Module are hollow Modules. Since \( T = Z_2 \oplus Z_8 \) is semilifting, then it is semihollow-lifting.

3. Let \( T \) be an \( \mathbb{R} \)-Module. The following statements are equivalent:

1. \( T \) is a semihollow Module.
2. For some proper Submodule \( H \) of \( T \), \( T/H \) is semihollow and \( H \triangleleft T \).
3. Every non-zero small factor Module of \( T \) is indecomposable.

**Proof:** Let \( H \leq T \). Then by Proposition 4, \( T/H \) is semihollow. 

**Proposition 6** Let \( T \) be an \( \mathbb{R} \)-Module. The following statements are equivalent:

1. \( T \) is a semihollow Module.
2. For some proper Submodule \( H \) of \( T \), \( T/H \) is semihollow and \( H \triangleleft T \).
3. Every non-zero small factor Module of \( T \) is indecomposable.

2 Suppose that \( H \leq T \) and \( T/H \) is semihollow. Let \( G \) be a proper Submodule of \( T \). Thus \( G + H \neq T \), then \( G + H \leq T/H \). Let \( T = G + F \), where \( F \subseteq T \). Then \( T/H = G + F/H \). Since \( G + H \leq T/H \), thus, \( T = F + H \). But \( H \leq T \). Hence \( T \) is semihollow Module.

3 Assume that \( T \) is a semihollow Module, and let \( H \) be a non-zero small factor Module of \( T \). Then by Proposition 4 we obtain \( T/H \) is semihollow. Hence by Proposition 5, \( T \) is indecomposable.

**Examples**

1. Consider the Module \( T = Z_2 \oplus Z_4 \), it is clear that, \( Z_2 \) and \( Z_4 \) as \( Z \)-Module are hollow Modules. Since \( T = Z_2 \oplus Z_4 \) is semilifting, then it is semihollow-lifting.

2. Assume that \( T \) has a semihollow factor Module. Thus there exists a proper Submodule \( H \) of \( T \) such that \( T/H \) is hollow. But \( T \) is semihollow-lifting, there is a direct summand \( F \).
of T such that $F \subseteq_{sc} H$ in T. Since T is indecomposable Module, therefore $F = 0$, hence $H \vartriangleleft_{s} T$. Then by proposition 6, T is semihollow.

2⇒1 Clear.

Now, the following example shows that a semihollow-lifting Module need not be semilifting.

**Example 8** Let T be an indecomposable $\mathbb{R}$-Module which has no semihollow factor Module. It is easy to show that T is semihollow-lifting. Claim that T is not semilifting. To show that, Assume that T is semilifting and let H be a proper Submodule of T. But T is semilifting, thus there exists a Submodule F of T such that $F \subseteq_{sc} H$ in T and $T = G \oplus G_1$ for some $G_1 \subseteq T$. Since T is indecomposable Module, thus $G = 0$ and hence $H \vartriangleleft_{s} T$. Then T is semihollow which is a contradiction.

A proper Submodule H of an $\mathbb{R}$-Module T is called a maximal Submodule in T, if whenever G is a Submodule of T with $H \subseteq G$, then $G = T$.[9].

**Proposition 9** Let T be an indecomposable semihollow-lifting Module, If T has a maximal Submodule, then it is unique.

**Proof:** Let H be a maximal Submodule of T. Suppose that T has another maximal Submodule F which is different from H, thus $T = H + F$. By[7], $T_{\mathbb{R}}$ is a simple Module and hence hollow. But T is semihollow-lifting Module, thus there is a direct summand D of T such that $D \subseteq_{sc} H$ in T. Since T is indecomposable Module thus $D = 0$, hence $H \vartriangleleft_{s} T$, this implies that $T = F$ that is a contradiction, then T has a unique maximal Submodule.

**Proposition 10** Let T be a semihollow-lifting Module, and let H and F be Submodules of T such that $T_{\mathbb{R}}$ hollow and $T = H + F$, then there exists a direct summand D of T such that $D \subseteq_{sc} H$ in T. Since T is indecomposable Module there exists $D = 0$, hence $H \vartriangleleft_{s} T$, this implies that $T = F$ that is a contradiction, then T has a unique maximal Submodule.

**Proof:** Let H and F be Submodules of T such that $T_{\mathbb{R}}$ hollow. But T is semihollow-lifting Module, there exists a direct summand D of T such that $D \subseteq_{sc} F$ in T. Since $T = H + F$, thus

$T_{\mathbb{R}} = \frac{H + F}{D} = \frac{H + A}{D} + \frac{K}{D}$. Since $D \subseteq_{sc} F$ in T, then $T_{\mathbb{R}} = \frac{H + A}{D}$. This implies that $T = H + D$.

A Submodule H of an $\mathbb{R}$-Module T is called semiclosed of T ($H \subseteq_{sc} T$), if $\frac{H}{F} \subseteq_{s} T$ implies that $H = F$ for all $F \subseteq T$ contained in $H_{[5]}$.

Let H and D be Submodules of a Module T, H is called semisupplement of D in T if $T = H + D$ and $H \cap D \vartriangleleft_{s} H$.

An $\mathbb{R}$-Module T is called a semisupplemented Module, if every Submodule of T has a semisupplement in T[6].

An $\mathbb{R}$-Module T is called an amply semisupplemented Module, if for any two Submodules H and F of M with $H + F = T$, F contains a semisupplement of H in T[6].

Let T be an $\mathbb{R}$-Module, and let H be a Submodule of T. A Submodule F of H is called a semiclosure Submodule of H in T, if F is a semicoessential Submodule of H in T and semiclosed of T. That is, $H/F \subseteq_{s} T/F$ and whenever $D \subseteq F$ with $F/D \subseteq_{s} T/D$ implies $D = [F/5]$.

**Proposition 11** Let T be an amply semisupplemented Module. Then every Submodule of T has a semiclosure Submodule.

**Proof:** Let H be a Submodule of T. But T is amply semisupplemented, thus there exists a Submodule F of T such that K is semiminal with the property $T = H + F$. But T is amply semisupplemented there exists a Submodule G of T such that $G \subseteq H$ and $T = G + F$ and $G \cap F \subseteq_{s} G$. Claim that G is a semiclosure Submodule of H in T. We must show that $G \subseteq_{sc} H$ in T. Let $G \subseteq E \subseteq T$ such that $\frac{H}{G} = \frac{E}{G}$. Thus $H + E = T$. By modular law, $E = G \cap (G + K) = G + (E \cap F)$, thus $T = H + G + (E \cap F)$. This implies that $T = H + (E \cap F)$. By minimality of F we get $F = E \cap F$. So $T = G + (E \cap F) = G + E$, then $T = E$, thus $G \subseteq_{sc} H$ in T.

**Proposition 12** Let T be an $\mathbb{R}$-Module, and let F and H be Submodules of T such that $F \subseteq H \subseteq T$, if $F \subseteq_{sc} H$ in T and $T_{\mathbb{R}}$ semihollow Module thus, $\frac{T}{F}$ semihollow Module.

**Proof:** Assume that $F \subseteq_{sc} H$ in T and $T_{\mathbb{R}}$ is semihollow Module. By the (third isomorphism theorem), $\frac{T}{H} \approx \frac{F}{T}$. But $\frac{T}{F}$ is semihollow and $F \subseteq_{sc} H$ in T, thus by proposition 6, $\frac{T}{F}$ is semihollow.

**Proposition 13** Let T be a semihollow-lifting Module, then every semiclosed Submodule F of T with $T_{\mathbb{R}}$ hollow is a direct summand of T.

**Proof:** Assume that T is a semihollow-lifting Module, and let F be a semiclosed Submodule in T such that $T_{\mathbb{R}}$ hollow. But T is semihollow-lifting, thus there is a direct summand H of T such that $H \subseteq_{sc} F$ in T. Since F is semiclosed in T, thus $F = H$. Then F is a direct summand of T.
Proposition 14 Let $T$ be a amply semisupplemented Module and every semiclosed Submodule $F$ of $T$ with $\frac{T}{F}$, $\frac{H}{F}$, $\frac{T}{H}$, $\frac{H}{T}$ hollow is a direct summand of $T$ then $T$ is semihollow-lifting Module.

**Proof:** Assume that $T$ is amply semisupplemented Module and every semiclosed Submodule $F$ of $T$ with $\frac{T}{F}$, $\frac{H}{F}$, $\frac{T}{H}$, $\frac{H}{T}$ hollow is a direct summand. We want to prove that $T$ is semihollow-lifting, let $H$ be a Submodule of $T$ with $\frac{H}{T}$ hollow. Thus by proposition 11, $H$ has a semiclosure Submodule say $F$ in $T$. Then $F \subseteq H$ in $T$ and $F \subseteq \frac{T}{H}$. But $\frac{H}{T}$ is hollow and hence semihollow, thus by proposition 12, $\frac{T}{F}$ is semihollow. Then by our assumption, we get $F$ is a direct summand. Thus $T$ is semihollow-lifting.

An $\mathbb{R}$-Module $T$ is said to have (D3) if for every direct summands $F$ and $D$ of $T$ with $T = F + D$, $F \cap D$ is a direct summand of $T[10]$.

An $\mathbb{R}$-Module $T$ is called a weakly semisupplemented Module, if for each Submodule $H$ of $T$, there exists a Submodule $F$ of $T$ such that $T = H + F$ and $F \cap H \ll_S T[6]$.

**Proposition 15** Let $T$ be a Module and $H \subseteq T$. Consider the following condition:
1. $H$ is a semisupplement Submodule of $T$.
2. $H$ is a semiclosed Submodule of $T$.
3. For all $F \subseteq T$ with $F \subseteq H$, $F \ll_S T$ implies $F \ll_S H$.

Then $1 \Rightarrow 2 \Rightarrow 3$ hold. If $T$ is a weakly semisupplemented Module, then $3 \Rightarrow 1$.

**Proof:** $1 \Rightarrow 2$ Let $H$ be a semisupplement of $D$ in $T$. Thus $T = H + D$ and $H$ is semiminimal with this property. Let $F \subseteq H \subseteq T$ such that $\frac{H}{F} \ll_S \frac{T}{F}$. Now $\frac{H}{F} = \frac{H + D}{F} = \frac{F + D + F}{F}$, $\frac{D + F}{F} = \frac{T}{F}$ and hence $T = D + F$. By minimality of $H$, we have $F = H$. Thus $H$ is semiclosed of $T$.

$2 \Rightarrow 3$ Let $G$ be a semiclosed Submodule of $T$. Let $F \subseteq G$ and $F \ll_S T$. Suppose that $G = F + D$, where $D \subseteq G$. It is enough to show that $\frac{G}{D} \ll_S \frac{D}{D}$.

To verify this, let $\frac{T}{D} = \frac{G}{D} + \frac{W}{D}$, where $D \subseteq W \subseteq T$. Thus $T = N + W = F + D + H = X + W$. Since $F \ll_S T$, then $T = W$ and hence $\frac{N}{D} \ll_S \frac{T}{D}$.

$3 \Rightarrow 1$ Assume that $T$ is a weakly semisupplemented Module. Then there exists a Submodule $D$ of $T$ such that $T = F + D$ and $F \cap D \ll_S T$. By our assumption, we get $F \cap D \ll_S F$. Thus $F$ is a semisupplement of $D$ in $T$.

The Submodules $F$ and $D$ are called mutual semisupplements in $\mathbb{R}$-Module $T$, if they are semisupplements of each other.

**Proposition 16** Let $T = H + F$ be a semihollow-lifting Module, where $H$ and $F$ are mutual semisupplements in $T$ with $\frac{T}{H}$ and $\frac{T}{F}$ are hollow Modules. If $M$ has (D3), then $T = H \oplus F$.

**Proof:** Let $H$ and $F$ be two Submodules of $T$ which are mutual semisupplements in $T$, with $\frac{T}{H}$ and $\frac{T}{F}$ are hollow Modules. Then by proposition 15, $H$ and $K$ are semiclosed Submodules of $T$. Since $T$ is semihollow-lifting, therefore by proposition 13, $H$ and $F$ are direct summands of $T$. But $T = H + F$ and $T$ has (D3), thus $H \cap F$ is a direct summand of $H$ so $H = (H \cap F) \oplus D$, for some $D \subseteq T$. But $F$ is a semisupplement of $H$ thus $H \cap F \ll_F H$ and hence $H \cap F \ll_S T$. Then $T = D$ and $H \cap F = 0$. Then we get $T = H \oplus F$.

The following proposition gives a condition under which a direct summand of a semihollow-lifting Module is semihollow-lifting.

**Proposition 17** Let $T$ be a semihollow-lifting Module having (D3). Then every direct summand of $M$ is semihollow-lifting.

Let $T$ be an $\mathbb{R}$-Module. A Submodule $D$ of $T$ is called a fully invariant Submodule if $h(D) \subseteq D$, for every $h \in Hom(T,T)[9]$.

**Lemma 18**[11] Let $T$ be an $\mathbb{R}$-Module. If $T = T_1 \oplus T_2$, then $\frac{T}{D} = \frac{T_1}{D} \oplus \frac{T_2}{D}$ for every fully invariant Submodule $D$ of $T$.

The following proposition gives a condition under which a factor of a semihollow-lifting Module is semihollow-lifting.

**Proposition 19** Let $T$ be an $\mathbb{R}$-Module. If $T$ is a semihollow-lifting module, then $\frac{T}{E}$ is semihollow-lifting for every fully invariant Submodule $E$ of $T$.

**Proof:** Let $\frac{H}{E}$ be a Submodule of $\frac{T}{E}$ such that $\frac{T}{E}$ is hollow. Thus by (third isomorphism theorem), $\frac{T}{E} \cong \frac{H}{E}$ is hollow. But $T$ is a semihollow-lifting Module, thus there exists a Submodule $F$ of $T$ such that $F \subseteq H$ in $T$ and $T = F \oplus F'$, for some $F'$ \subseteq T. It is clear that $F + E \subseteq H$ and hence $\frac{F + E}{E} \subseteq \frac{H}{E}$. Let $f : \frac{T}{F} \rightarrow \frac{T}{F + E}$ be a map defined by $f(m + F) = m + (F + E)$, for all $m \in T$. Clearly $f$ is an epimorphism. But $F \subseteq H$ in $T$, thus by[6], $f(\frac{H}{F}) \ll_S \frac{T}{F + E}$ and hence $F + E \subseteq H$ in $T$. Then by (third isomorphism theorem), we get $\frac{T}{F + E} \subseteq \frac{H}{E}$ in $\frac{T}{E}$.
By Lemma 18, \( \frac{T}{E} = \frac{F \oplus F'}{E} = \frac{F + E}{E} \oplus \frac{F' + E}{E} \). Thus \( \frac{T}{E} \) is a direct summand of \( \frac{F}{E} \). Then \( \frac{T}{E} \) is semihollow-lifting.

In proposition 19, if \( T \) is a semihollow-lifting Module and \( E \) is not fully invariant Submodule of \( T \), then \( \frac{T}{E} \) need not be semihollow-lifting.

**Example 20** Consider the \( \mathbb{Z} \)-Module \( T = \mathbb{Z} \oplus \mathbb{Z} \), it is known that \( T \) is hollow-lifting[11] and hence semihollow-lifting. Let \( E = \mathbb{Z} \oplus \mathbb{Z} \) be a Submodule of \( T \), thus \( \frac{T}{E} \) is not semihollow-lifting. To see that: \( \frac{T}{E} = \frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z} \oplus \mathbb{Z}} \cong \frac{\mathbb{Z}}{\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\mathbb{Z}} \), then \( \frac{T}{E} \) is not semihollow-lifting by Example 3.

An \( \mathbb{R} \)-Module \( T \) is called a duo-Module if every Submodule of \( T \) is fully invariant[11].

The following proposition gives a condition under which a direct summand of a semihollow-lifting Module is semihollow-lifting.

**Proposition 21** Let \( T \) be a duo semihollow-lifting Module. Then every direct summand of \( T \) is a semihollow-lifting.

**Proof:** Clear, by proposition 19.

**Theorem 22** Let \( T \) be a non-zero indecomposable Module over a commutative ring \( \mathbb{R} \). Then the following are equivalent:

1. \( T \) is semihollow-lifting.
2. \( T \) is semilifting.
3. \( T \) is semihollow.

**Proof:** Clear.

**Lemma 23[7]** Let \( f : T \to H \) be an epimorphism of \( \mathbb{R} \)-Modules and \( T = F + D \), where \( F \) and \( D \) are Submodules of \( T \) then:

1. \( H = f(F) + f(D) \).
2. If \( \ker f = F \cap D \), then \( H = f(F) \oplus f(D) \).

**Proposition 24[12]** Epimorphic image of hollow Module is hollow.

**Proposition 25** Let \( f : T \to H \) be an epimorphism of \( \mathbb{R} \)-Modules. Let \( F \) and \( D \) be Submodules of \( T \) such that \( T = F + D \) and \( \ker f = F \cap D \). If \( H \) is a semihollow-lifting Module and \( D \) is hollow, then \( H = T_1 \oplus T_2 \), where \( T_1 \subseteq_{sce} f(F) \) in \( H \) and \( T_2 \) is hollow.

**Proof:** By lemma 23, we have \( H = f(K) \oplus f(D) \). Since \( D \) is hollow, then by proposition 24, \( f(D) \) is hollow. Thus by (second isomorphism theorem), \( \frac{H}{f(F)} \cong f(D) \). Since \( H \) is a semihollow-lifting Module, there exists a direct summand \( T_1 \) of \( H \) such that \( T_1 \subseteq_{sce} f(F) \) in \( H \). Thus \( H = T_1 \oplus T_2 \), where \( T_2 \subseteq H \). Now, \( \frac{H}{T_1} = \frac{f(K) \oplus f(D)}{T_1} = \frac{f(K)}{T_1} + \frac{f(D)}{T_1} \). This implies \( H = (D) \oplus T_1 \). By the (second isomorphism theorem), we get \( \frac{H}{T_1} \cong f(D) \) and \( \frac{H}{T_1} \cong T_2 \) and hence \( T_2 \cong f(D) \). Then \( T_2 \) is hollow.

**Conclusion**

In this paper, we give the concept of semihollow-lifting Module. Also, study basic properties of semihollow-lifting Modules.

1. Every Module which has no semihollow factor Module is semihollow-lifting.
2. Every semilifting Module is semihollow-lifting, in particular every semisimple or semihollow Module is semihollow-lifting.
3. Let \( F_1 \) and \( F_2 \) be hollow Modules. Then the following are equivalent for the module \( T = F_1 \oplus F_2 \):
   1. \( T \) is semihollow lifting Module.
   2. \( T \) is semilifting Module.
   3. \( T \) is indecomposable Module. The following statements are equivalent:
      1. \( T \) is a semihollow-lifting Module.
      2. \( T \) is semihollow or else \( T \) has no semihollow factor Modules.
      4. Give example shows that a semihollow-lifting Module need not be semilifting.
      5. Let \( T \) be a non-zero indecomposable Module, if \( M \) has a maximal Submodule, then it is unique.
      6. Let \( T \) be a semihollow-lifting Module, if \( M \) is a Submodule of \( T \) such that \( \frac{T}{F} \) hollow and \( T = H + F \), then there exists a direct summand \( D \) of \( T \) such that \( T = H + D \) and \( D \subseteq_{sce} F \) in \( T \).
5. Let \( T \) be a semihollow-lifting Module, then every semiclosed Submodule \( F \) of \( T \) with \( \frac{T}{F} \) hollow is a direct summand of \( T \).
6. Let \( T \) be a semihollow-lifting Module, and let \( H \) and \( F \) be Submodules of \( T \) such that \( \frac{T}{F} \) hollow and \( T = H + F \), then there exists a direct summand \( D \) of \( T \) such that \( T = H + D \) and \( D \subseteq_{sce} F \) in \( T \).
7. Let \( T \) be a semihollow-lifting Module, and let \( H \) and \( F \) be Submodules of \( T \) such that \( \frac{T}{F} \) hollow and \( T = H + F \), then there exists a direct summand \( D \) of \( T \) such that \( T = H + D \) and \( D \subseteq_{sce} F \) in \( T \).
8. Let \( T \) be a semihollow-lifting Module, then every semiclosed Submodule \( F \) of \( T \) with \( \frac{T}{F} \) hollow is a direct summand of \( T \).

10. Let \( T \) be a semihollow-lifting Module having \( (D) \). Then every direct summand of \( T \) is semihollow-lifting.
11. Let \( T \) be an \( \mathbb{R} \)-Module. If \( T \) is a semihollow-lifting Module, then \( \frac{T}{D} \) is semihollow-lifting for every fully invariant Submodule \( D \) of \( T \).
12. If \( T \) is a semihollow-lifting Module and \( D \) is not fully invariant Submodule of \( T \), then \( \frac{T}{D} \) need not be semihollow-lifting.
13. Let \( T \) be a duo semihollow-lifting Module. Then every direct summand of \( T \) is a semihollow-lifting.
14) Let $T$ be a non-zero indecomposable Module over a commutative ring $\mathbb{R}$. Then the following are equivalent:
1. $T$ is semihollow-lifting.
2. $T$ is semilifting.
3. $T$ is semihollow.

15) Let $f: T \rightarrow H$ be an epimorphism of $\mathbb{R}$-Modules. Let $F$ and $D$ be Submodules of $T$ such that $T = F + D$ and $\ker f = F \cap D$. If $H$ is a semihollow-lifting Module and $D$ is hollow, then $H = T_1 \oplus T_2$, where $T_1 \subseteq_{sc} f(F)$ in $H$ and $T_2$ is hollow.

REFERENCES


