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Unitary and commutative rings and their relations with ideals

Los anillos unitarios y conmutativos y sus relaciones con los ideales.

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ABSTRACT/ Based on the fact from ring theory, that the ideal of the direct product of commutative-unitary-rings is a decomposition process to ideals of their factors. The present paper will study this fact for the commutative rings, to find necessary and sufficient conditions required for achievement, i.e., in the present paper, we will seek all type conditions required for the relation between commutative rings and ideals.
Keywords: Unitary rings, commutative rings, ideals
RESUMEN/ Basado en el hecho de la teoría de los anillos, que el ideal del producto directo de los anillos conmutativos unitarios es un proceso de descomposición de los ideales de sus factores. El presente trabajo estudiará este hecho para los anillos conmutativos, para encontrar las condiciones necesarias y suficientes requeridas para el logro, es decir, en el presente documento, buscaremos todas las condiciones de tipo requeridas para la relación entre los anillos conmutativos y los ideales.
Palabras clave: anillos unitarios, anillos conmutativos, ideales.

Introduction

Algebra is a branch of mathematics, the name of algebra came from the book of mathematician, astronomer and traveler Muhammad ibn Musa al-Khwarizmi (A branch of mathematics based on the replacement of symbols of unknown numbers or information) **[1-3]**. Algebra is one of the three basic branches in mathematics in addition to mathematical engineering and mathematical analysis and numerology, this science is concerned with the study of algebraic structures and symmetries, and relationships, quantities, and algebra is a broader and more comprehensive concept than arithmetic or elementary algebra **[4-6]**. In the universe, therefore, it is one of the essential bases of the methods of proof and in abstract algebra. The theory of rings is the study of rings and algebraic structures in which addition and plural processes are known to have properties similar to integers **[7-8]**. Based on the fact from ring theory, that the ideal of the direct product of commutative-unitary-rings is a decomposition process to ideals of their factors. The present paper will study this fact for the commutative rings, to find necessary and sufficient conditions required for achievement, i.e., in the present paper, we will seek all type conditions required for the relation between commutative rings and ideals **[9]**.

General mathematical structure

Let us start by the ideal of the ring R_{ring} and let:

$$R_{ring} = (R,+,.) \subset I \quad (1)$$

Under the condition $\xi, \zeta \subset I$, therefore,

$$\xi,-\zeta \in I,\ \forall r \in R \quad (2)$$

And

$$\xi r, r\xi \in I \forall r \in R \quad (3)$$

The R-T affected directly by the ideals, where the homomorphisms-kernels its-self are I_{ideal} and the R_{ring} can be analyzed to factors by making use of I_{ideal}.

From the basis, one can remained the fact that both R_{ring}, I_{ideal}, and C_{congl} all are $O\text{-}T\text{-}O$.

Therefore;

If $R_{ring} = (R,+,.)$ is R_{ring}, then

$I_{ideal}LR_{ring} : (I_{ideal}LR_{ring}, \subseteq)$ and

$C_{congl}LR_{ring} := (C_{congl}LR_{ring}, \subseteq)$ are $ISO_{morphicl}$

Therefore; $I_{ideal}LR_{ring} : (I_{ideal}LR_{ring}, \subseteq)$ is MBL and the MBL have $\{0\}$ as least element and the greatest-element of R_{ring}.

The supermum and infimum of I_{ideal} takes the form:

$$I_{ideal1} \vee I_{ideal2} = I_{ideal1} + I_{ideal2} \quad (4)$$

And

$$I_{ideal1} \wedge I_{ideal2} = I_{ideal1} \cap I_{ideal2} \quad (5)$$

Assume that we have two rings as follow:

$$R_{ring_1} = (R_1,+,.) \quad (6\text{-}1)$$

$$R_{ring_2} = (R_2,+,.) \quad (6\text{-}2)$$

With

$$I_1 \in I_{ideal}LR_{ring_1} \quad (6\text{-}3)$$

$$I_2 \in I_{ideal}LR_{ring_2} \quad (6\text{-}4)$$

Then

$$I_1 \times I_2 \in I_{ideal}L(R_{ring_1} \times R_{ring_2}) \quad (7)$$

Conversely, if

$$I \in I_{ideal}L(R_{ring_1} \times R_{ring_2}) \quad (8)$$

It does not need existence of equations (6-3) and (6-4), with

$$I_1 \times I_2 = I \quad (9)$$

As a result, if I_1, I_2 are not exist, therefore I will be skew. One can say that the direct-product of so many rings of finite form has direct-decomposable-ideals if $\forall I_{ideal}$ of the product is a direct-product of I_{ideal} having the corresponding-factors.

Condition of skewness of two rings

Assume that we have two-commutative-rings $R_{Com\ ring_1}$ and $R_{Com\ ring_2}$, in what follow, the conditions for $R_{Com\ ring_1} \times R_{Com\ ring_2}$ has no skew-ideals. The direct-product rings of finite number has a direct decomposable ideals if each ideal of the product will be a direct-ideals-product having the same factors.

Assume that we have two sets, namely $M_1 \,\&\, M_2$ and $\Pi_j,\ j = 1,2$ be the projections from $M_1 \times M_2$ onto M_j, now the ideal of $R_{Com\ ring_1} \times R_{Com\ ring_2}$ is a direct decomposition iff:

$$\Pi_1(I) \times \Pi_2(I) = I$$

$$\Leftrightarrow$$

$$\Pi_1(I) \times \Pi_2(I) \subseteq I \quad (10)$$

Theorem

Assume that $R_{ring_1} = (R_1,+,.)$ and $R_{ring_2} = (R_2,+,.)$ be two rings and $I \in I_{ideal}(R_{Com\ ring_1} \times R_{Com\ ring_2})$, then:

- I is of direct decomposition
  $(R_{ring_1} \times \{0\}) \cap ((\{0\} \times R_{ring_2}) + I) \subseteq I$
  And
- $((R_{ring_1} \times \{0\}) + I) \cap (\{0\} \times R_{ring_2}) \subseteq I \quad (11)$
  $(\xi, \zeta) \in I$
  $\Rightarrow$
- If $(\xi,0),(0,\zeta) \in I \quad (12)$

$$((R_{ring_1} \times \{0\}) + I) \cap ((\{0\} \times R_{ring_2}) + I) \subseteq I \quad (13)$$

Proof

Let us assume that:

$$I = I_1 \times I_2 \quad (14)$$

Then

$$\left(R_{ring_1} \times \{0\}\right) \cap \left(\left(\{0\} \times R_{ring_2}\right) + I\right) = \left(R_{ring_1} \times \{0\}\right) \cap \left(\{0\} \times R_{ring_2}\right) + \left(I_1 \times I_2\right)$$
$$= \left(R_{ring_1} \times \{0\}\right) \cap \left(\{0\} \times R_{ring_2}\right) = I_1 \times \{0\} \subseteq I$$

&

$$\left(\left(R_{ring_1} \times \{0\}\right) + I\right) \cap \left(\{0\} \times R_{ring_2}\right) = \left(R_{ring_1} \times \{0\}\right) + \left(I_1 \times I_2\right) \cap \{0\} \times R_{ring_2}$$
$$= \left(R_{ring_1} \times I_1\right) \cap \left(\{0\} \times R_{ring_2}\right) = \{0\} \times I_2 \subseteq I$$

(15)

Now, if we assume that:

$$(\xi, \zeta) \in I \quad (16)$$

Then

$$(\xi, 0) \in \left(R_{ring_1} \times \{0\}\right) \cap \left(\left(\{0\} \times R_{ring_2}\right) + I\right) \subseteq I$$

$$(0, \zeta) \in \left(\left(R_{ring_1} \times \{0\}\right) + I\right) \cap \left(\{0\} \times R_{ring_2}\right) \subseteq I \quad (17)$$

Assume that

$$(\xi, \zeta) \in \Pi_1(I) \times \Pi_2(I) \quad (18)$$

Then $\exists$ some $(\hbar, \lambda) \in R_{ring_1}, R_{ring_2}$ with

$$(\xi, \lambda), (\hbar, \zeta) \in I$$

Therefore

$$(\xi, 0) \& (0, \zeta) \in I \quad (19)$$

This leads to

$$(\xi, \zeta) = (\xi, 0) + (0, \zeta) \in I$$

Finally, let us assume that

$$I = I_1 \times I_2 \quad (20)$$

Therefore;

$$\left(\left(R_{ring_1} \times \{0\}\right) + I\right) \cap \left(\left(\{0\} \times R_{ring_2}\right) + I\right) =$$
$$= \left(R_{ring_1} \times \{0\}\right) + \left(I_1 \times I_2\right) \cap \left(\left(\{0\} \times R_{ring_2}\right) + \left(I_1 \times I_2\right)\right) =$$
$$= \left(R_{ring_1} \times I_2\right) \cap \left(I_1 \times R_{ring_2}\right) = \left(I_1 \times I_2\right) = I \quad (21)$$

Theorem

Assume that $R_{ring_1} = (R_1, +, .)$ and $R_{ring_2} = (R_2, +, .)$ be two rings, therefore, $I_{ideal}\left(R_{Com\ ring_1} \times R_{Com\ ring_2}\right)$ is distributive iff $\forall\left(R_{Com\ ring_1} \times R_{Com\ ring_2}\right)$ is of direct decomposition and $I_{ideal} R_{Com\ ring_1} \& I_{ideal} R_{Com\ ring_2}$ are distributive.

Theorem

Assume that $\xi, \zeta, \hbar, \lambda \in Z$ and $\hbar | \xi \& \lambda | \zeta$, also the $I_{ideal}(\xi, \zeta)$ of $\hbar Z \times \lambda Z$ governed by (ξ, ζ), therefore, one can say that $I_{ideal}(\xi, \zeta)$ is of direct decomposition iff one of the following cases satisfied $\xi = 0$ or $\zeta = 0$

Proof

Let us start the proof by assuming the following relation:

$$(\xi, \zeta)Z := (\xi\aleph, \zeta\aleph) | \aleph \in Z \quad (22)$$

It is clear

$$I(\xi, \zeta) = (\xi, \zeta)Z + (\xi\hbar Z \times \zeta\lambda Z) \quad (23)$$

$$\Pi_1(I(\xi, \zeta)) = \xi Z$$

&

$$\Pi_2(I(\xi, \zeta)) = \zeta Z \quad (24)$$

Both equations are of direct decomposition of $I(\xi, \zeta)$, and they are equivalent to:

$$\xi Z \times \zeta Z \in I(\xi, \zeta) \quad (25)$$

Now in case of $\xi = 0$ or $\zeta = 0$, then it will be so clear that $\xi Z \times \zeta Z \in I(\xi, \zeta)$

Theorem

Assume that we have vR_{ring} therefore, it has a direct decomposition of I_{ideal}, and recalling CR_{ring} defined as:

$$CR_{ring} : \Im(\aleph) = (\Im(\aleph), +, .) \quad (26)$$

$$I(\aleph, \aleph) = (N(\aleph, \aleph) + (\aleph, \aleph)(\Re(\aleph), \Phi(\aleph))) \quad (27)$$

And

$$(\Re(\aleph), \Phi(\aleph)) \in \Im(\aleph) \times \Im(\aleph)$$
$$= N\aleph + \aleph\Re(\aleph), N\aleph + \aleph\Phi(\aleph) | N \in Z \& \Re(\aleph), \Phi(\aleph) \in \Im(\aleph) \quad (28)$$

Since $I(\aleph, \aleph)$ is of a direct decomposition

And

$$(0,0), (\aleph, \aleph) \in I(\aleph, \aleph) \quad (30)$$

Therefore,

$$(\aleph, 0) \in I(\aleph, \aleph) \exists N \in Z, \Re(\aleph), \Phi(\aleph) \in \Im(\aleph) \quad (31)$$

With

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$$\left(N\aleph + \aleph\Re(\aleph) + \aleph\Phi(\aleph)\right) = (\aleph, 0)$$

$$\Rightarrow$$

$$\aleph\,\tau(\aleph) = \aleph$$

With

$$\tau(\aleph) := \Re(\aleph) - \Phi(\aleph) \qquad (32)$$

Conclusion

In the current research paper, and by making use of some facts from ring-theory, we proved that I_{ideal} direct-product of commutative-unitary-rings is of decompose process to I_{ideal} of their factors. The present paper also studied some facts for the commutative-rings, and found some necessary and sufficient conditions required for achievement, i.e., looked for all type conditions required for the relation between commutative-rings and ideals. It is concluded from the present paper that the ideal of the direct-product of commutative-unitary-rings is of a decompose process to ideals of their factors. Also the paper studied the fact of the commutative-rings, and found the necessary and the sufficient conditions required for achievement the relation between commutative rings and ideals.

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Notations

R_{ring} : Ring notation throughout the text

R-T: Rings-theory

I_{ideal} : Ideals

C_{congl} : Congruence

$O\text{-}T\text{-}O$: One-to-one

$I_{ideal}L$: Ideal lattice

$C_{congl}L$: Congruence lattice

$ISO_{morphicl}$: Isomorphic

MBL : Modular bounded lattice

vR_{ring} Variety of commutative rings

CR_{ring} Free commutative ring

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